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BOOK II.

BLAIKIE AND THOMSON

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A TEXT-BOOK OF
GEOMETRICAL DEDUCTIONS

BOOK II.

Corresponding to Euclid, Book II.

WITH MISCELLANEOUS DEDUCTIONS FROM BOOKS
I. AND II.

BY

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PREFACE

THE object of this treatise is to afford a systematic course of training in the art of solving Geometrical Deductions or Riders. With this view it is divided into sections, each section consisting of three parts. There is first a deduction worked out in full, which is intended to serve as a model for the student. This is followed by a number of similar deductions, which are to be written out by the student, the figure being given in each case, and such hints regarding the mode of solution as experience shows are required by beginners. Lastly, each section contains some deductions to be accomplished without this aid, no figures or assistance being given except an occasional reference to the proposition on which the proof depends, or to a previous example.

As a rule, it is desirable that the proofs should depend upon propositions of Euclid, and not upon previous examples, the only exception being in the case of certain standard theorems which are indicated in the text.

For convenience of reference, especially in the case of those who have used text-books other than Euclid's, the enunciations of Euclid's propositions are given in an Appendix.

It is not necessary, and perhaps not desirable, that on his first reading the student should work through every example in each section. He should in each case, however, write out a sufficient number to insure his mastery of the principles involved; the others will be found useful when he comes to revise.

The exercises have been gathered from all available sources, including examination papers and geometrical text-books, English and foreign.

The authors acknowledge valuable suggestions and assistance from Messrs. Butters, Clark, and Walker, Heriot's Hospital School; Mr. R. F. Davis; the Rev. W. F. Failes, Westminster School; Mr. Hayward, Harrow School; Mr. Macdonald, Daniel Stewart's College; Dr. Mackay, Edinburgh Academy; Rev. J. J. Milne; Dr. Muir, Glasgow High School; Professor Raitt, Glasgow Technical College; Mr. Robertson and Mr. Mackay, Edinburgh Ladies' College; Professor Scott Lang, University of St. Andrews; Rev. G. Style and Mr. Wynne-Edwards, Giggleswick School; Mr. Tucker, University College School; Dr. Kolbe, Cape Town, and other friends.

Additional parts, corresponding to the remaining books of Euclid, are in preparation.

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SYMBOLS.

∴	signifies	*therefore.*
$=$	„	*is equal to.*
$\equiv$	„	*is equal in all respects to, is congruent with,* or *is identically equal to.*
$\perp$	„	*is perpendicular to* or *perpendicular to.*
$\parallel$	„	*is parallel to* or *parallel to.*
$>$	„	*is greater than.*
$<$	„	*is less than.*
$+$	„	*together with.*
$-$	„	*diminished by.*
△	„	*triangle.*
∠	„	*angle.*
rt. ∠	„	*right angle.*
▱	„	*parallelogram.*
AB^2	„	*square on* AB.
AB·CD	„	*rectangle contained by* AB *and* CD.
AB~CD	„	*the difference between* AB *and* CD.

BOOK II.

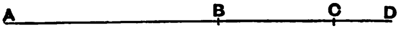
CHAPTER I.

THEOREMS.

§ 31. (*Bookwork*, EUCLID, II. 1-3.)

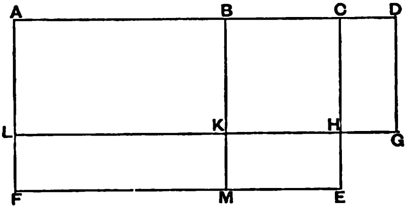
1. If A, B, C, D be four points in order in the same straight line, the rectangle AC·BD shall be equal to the sum of the rectangles AD·BC and AB·CD.

(*A Standard Theorem.*)



$$\begin{aligned} AC\cdot BD &= AC\cdot BC + AC\cdot CD, && \text{[Euc. II. 1.} \\ &= AC\cdot BC + AB\cdot CD + BC\cdot CD, && \text{[,,} \\ &= AC\cdot BC + CD\cdot BC + AB\cdot CD, && \text{[By Transposition.} \\ &= AD\cdot BC + AB\cdot CD. && \text{[Euc. II. 1.} \end{aligned}$$

2. Prove the above theorem by means of a Geometrical Construction.



Draw CE and DG $\perp$ AD and equal to BD and BC respectively.

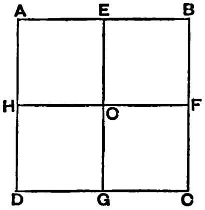
Complete the rectangles AG, AE and AM.

Prove that CG and KE are rectangles contained by equal lines, and are therefore equal.

Hence show that AE=AG+LM, and deduce the required result.

3. The square on any straight line is four times the square on half the line.

(A Standard Theorem.)

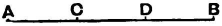


Use Euc. II. 1, to show that

$$AB^2=AB\cdot AE+AB\cdot EB, \text{ etc.};$$

or prove by Geometrical Construction. See also the proof given in § 20, Ex. 3.

4. The square on any straight line is nine times the square on one-third of the line.



Use Euc. II. 1, or prove by Geometrical Construction.

5. If AB be divided in D so that $AD=2\,DB$, show that

$$4AB^2=9\,AD^2.$$

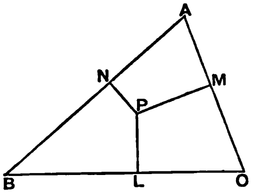
Use the methods of Ex. 3 and 4.

6. If AB be trisected in C and D, show that

$$AB^2=AD^2+BC^2+CD^2.$$

7. If P be any point within a △ ABC and PL, PM, PN be drawn $\perp$ BC, CA, AB; show that

$$BC\cdot BL+CA\cdot CM+AB\cdot AN=BC\cdot LC+CA\cdot MA+AB\cdot NB.$$



Use Euc. II. 3 to show that

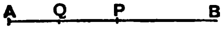
$$BC\cdot BL=BL^2+BL\cdot LC, \text{ etc.}$$

$$BC\cdot LC=BL\cdot LC+LC^2, \text{ etc.}$$

and apply § 20, Ex. 1.

8. If P be any point in AB, and Q be the mid-point of AP, show that $AB\cdot AQ=2\,AQ^2+QP\cdot PB.$

Use Euc. II. 3 to show that



$$AB\cdot AQ=AQ^2+AQ\cdot BQ,$$

$$=AQ^2+QP\cdot QB=\text{etc.}$$

9. The hypotenuse BC of a right-angled △ ABC is divided in D, so that $BC\cdot BD=AB^2$; show that $BC\cdot DC=AC^2$.

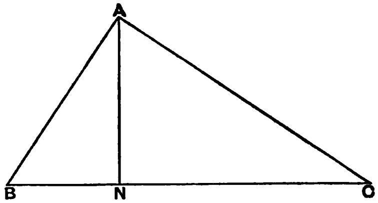
§ 32. (*Bookwork*, EUCLID, II. 1-4.)

DEFINITION.—*If a point be taken in a straight line, its distances from the extremities of the line are called* **segments** *of the line, and the point is called a* **Point of Internal Section.**

1. In a right-angled triangle, the square on the altitude from the right angle is equal to the rectangle contained by the segments into which the altitude divides the hypotenuse; and the square on either of the sides which contain the right angle is equal to the rectangle contained by the hypotenuse and its adjacent segment. (*A Standard Theorem.*)

Let ABC be a triangle, right-angled at A, and let AN be the altitude from A, it is required to prove that

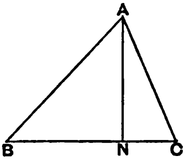
(1) $AN^2 = BN \cdot NC$,
(2) $AB^2 = BC \cdot BN$,
and $AC^2 = BC \cdot NC$.



(1) $AB^2 = BN^2 + AN^2$. [Euc. I. 47.
Similarly $AC^2 = NC^2 + AN^2$, [,,
$BC^2 = AB^2 + AC^2$; [,,
$\therefore BC^2 = BN^2 + NC^2 + 2AN^2$.
But $BC^2 = BN^2 + NC^2 + 2BN \cdot NC$; [Euc. II. 4.
$\therefore AN^2 = BN \cdot NC$.
(2) As before, $AB^2 = BN^2 + AN^2$;
$= BN^2 + BN \cdot NC$;
$= BN \cdot BC$; [Euc. II. 3.
$= BC \cdot BN$.
Similarly $AC^2 = BC \cdot NC$.

2. Prove the above theorem by means of the geometrical construction employed by Euclid in proving Euc. I. 47.

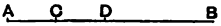
3. In any triangle in which the altitude from the vertex falls within the base, the square on the base, together with twice the square on the altitude, is equal to the squares on the sides, together with twice the rectangle contained by the segments of the base.



Use Euc. I. 47 and II. 4, to show that

$$BC^2+2AN^2=AB^2+AC^2+2BN\cdot NC.$$

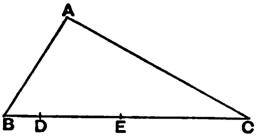
4. If a straight line be divided into three segments, the square on the whole line shall be equal to the squares on the segments, together with twice the rectangles contained by each pair of segments.



Consider AB as divided in C, and CB as divided in D, and apply Euc. II. 4 to each. Use also Euc. II. 1.

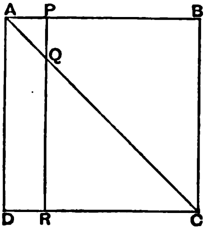
5. In the △ABC, ∠A is a right angle, and D, E are points in BC, such that BE=AB, DC=AC; show that

$$DE^2=2BD\cdot EC.$$



Obtain two expressions for BC^2 by means of Euc. I. 47, II. 4, and by the method of the previous Ex.

6. ABCD is a square, and through P a point in AB, PQR is drawn ⊥ AB, meeting AC in Q, and DC in R; show that AP·PB =PQ·QR=½AQ·QC.



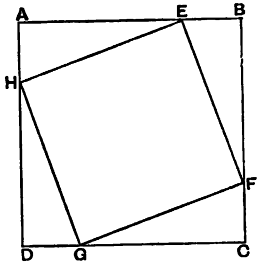
Apply Euc. II. 4 to AB and AC, etc.

7. Obtain a second proof of Euc. II. 4 from the theorem:—If, in the sides of a given square, four points be taken at equal distances from the angular points towards the same parts, the figure contained by the straight lines which join these points shall be a square. (§ 12, Ex. 18.)

Use Euc. I. 4, I. 13, and I. 32, to show that EFGH is a square.

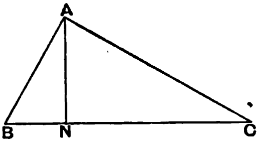
Use § 18, Ex. 5, to show that each of the triangles $=\frac{1}{2}AE \cdot EB$, and $\therefore AB^2 = EF^2 + 2AE \cdot EB$.

Apply Euc. I. 47.



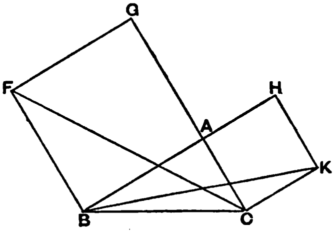
8. If, in a right-angled triangle, one of the sides which contain the right angle be equal to twice the other, the altitude from the right angle shall divide the hypotenuse into segments, one of which is four times the other.

Use Ex. 1 (2), and § 31, Ex. 3



9. ABC is a triangle, right-angled at A, and on AB and AC squares ABFG, ACKH are described; show that

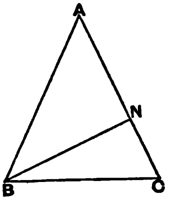
$$BK^2 + CF^2 = 3BC^2 + 4AB \cdot AC.$$



Use Euc. I. 47 and II. 4.

10. ABC is an isosceles triangle, having AB=AC, and BN is the altitude from B; show that

(1) $BN^2=2AN\cdot NC+NC^2$.
(2) $BC^2=2AC\cdot NC$.



(1) Use Euc. I. 47 and II. 4.

(2) Deduce this from (1) by Euc. II. 3.

11. If, in a triangle, the square on the altitude be equal to the rectangle contained by the segments into which the altitude divides the base internally, the angle from which the altitude is drawn shall be a right angle.

See Ex. 1 (1), to which this is a converse.

12. O is the centre of a square ABCD, the diagonal AOC is produced to E, so that CE=AB; show that $AE^2=2OE^2$.

Draw $EF \perp AB$, and prove $EF=OE$, etc.

13. ABC is a triangle, right-angled at A, D is the mid-point of BC, and from D, DE is drawn $\perp$ BC, meeting AB produced if necessary in E; show that $BC^2=2BA\cdot BE$.

Join EC, and use Euc. I. 47 and II. 4, etc.

14. AB is divided into five equal parts at C, D, E, F; show that $AB^2=AE^2+AF^2$.

15. Prove Ex. 4 by means of a geometrical construction similar to that used in Euc. II. 4.

16. A, B, C, D are four points in order in a straight line, and BC=CD; show that

$$AB^2+2AC\cdot CD=AC^2+CD^2.$$

17. A, B, C, D, E are five points in order in a straight line, and BC=CD; show that

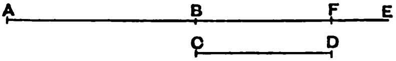
$$AC^2+BE\cdot DE=CE^2+AB\cdot AD.$$

§ 33. (*Bookwork*, Euclid, II. 1-6.)

1. The rectangle contained by the sum and the difference of two straight lines is equal to the difference between their squares. (*A Standard Theorem.*)

Let AB, CD be the two given straight lines, and let $AB > CD$, it is required to prove that

$$(AB+CD)\cdot(AB-CD)=AB^2-CD^2.$$



Produce AB to E, so that $BE=AB$.

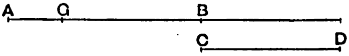
From BE cut off $BF=CD$.

$$AF\cdot FE+BF^2=AB^2. \qquad \text{[Euc. II. 5.}$$

$$\therefore AF\cdot FE=AB^2-BF^2,$$

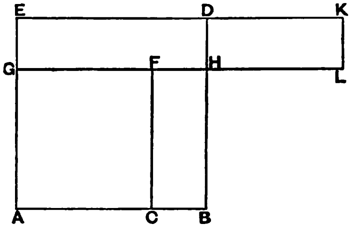
$$\text{or } (AB+CD)\cdot(AB-CD)=AB^2-CD^2.$$

2. Deduce the same theorem from Euc. II. 6.



Produce AB to F, so that $BF=CD$. From BA cut off $BG=CD$. Use Euc. II. 6.

3. Prove the same theorem by means of a geometrical construction.



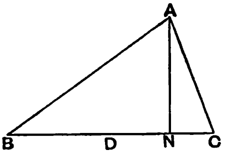
Let AB, AC be the given straight lines. Describe squares on them as in the figure, and produce ED to K, so that $DK=AC$. Produce GF and draw $KL \perp GF$.

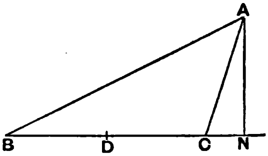
Prove $EL=EH+FB$, etc.

Observe that this is a general theorem, of which both Euc. II. 5 *and II.* 6 *are particular cases.*

4. The difference between the squares on two sides of a triangle is equal to twice the rectangle contained by the base and the segment of the base intercepted between its mid-point and the foot of the altitude.

(A Standard Theorem.)



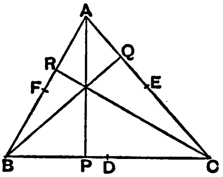


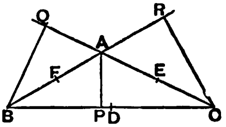
Use § 20, Ex. 5 and § 33, Ex. 1.

Show that $2DN = BN \sim NC$ in figure 1, and that $2DN = BN + CN$ in figure 2, and hence that $AB^2 \sim AC^2 = 2BC \cdot DN$.

5. In the △ ABC, D, E, F are the mid-points of BC, CA, AB, and AP, BQ, CR are the altitudes. If BC>CA>AB, show that

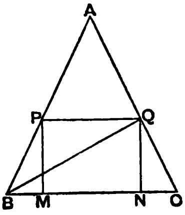
$$BC \cdot PD + AB \cdot RF = CA \cdot EQ.$$





As in the previous Ex., show $CA^2 - AB^2 = 2BC \cdot PD$, etc.

6. In the △ ABC, AB=AC, and PQ ∥ BC, meeting AB, AC in P, Q; show that $BQ^2 - CQ^2 = BC \cdot PQ$.



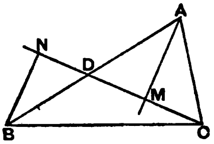
Draw $PM, QN \perp BC$.

Use § 20, Ex. 5 and § 33, Ex. 1.

Consider, also, the cases in which P is a point in AB, or BA, produced.

7. In the $\triangle$ ABC, AM bisects $\angle$ A, CM $\perp$ AM, and meets AB in D, and BN $\perp$ CD; show that $BC^2 - BD^2 = 4CM \cdot MN$.

Compare Ex. 4.



8. If a square and a rectangle have the same perimeter, show that the square has the greater area.

In the figure of Ex. 3, show that AD and EL have the same perimeter, but that AD=EL+AF.

9. In a right-angled triangle, the square on one of the sides which contain the right angle is equal to the rectangle contained by the sum and difference of the hypotenuse and the other side.

Use Ex. 1.

10. A, B, C, D are four points in order in a straight line, and AB=BC; show that $AD^2 - CD^2 = 2AC \cdot BD$.

Use Ex. 1.

11. A, B, C, D are four points in order in a straight line, and AB=BD; show that $AC^2 - CD^2 = 2AD \cdot BC$.

12. A, B, C are three points in a straight line, such that AB=BC, and P is any point in the line; show that $AP^2 \sim CP^2 = 4\, AB \cdot BP$.

13. ABC is an isosceles triangle, and D is the mid-point of the base BC; BC is produced to E, so that DE=AB; show that $AD^2 = BE \cdot CE$.

14. ABC is a triangle, right-angled at A, E is the mid-point of AC, and EM is drawn $\perp$ BC; show that $BM^2 - CM^2 = AB^2$.

15. A, B, C, D, E, F are six points in order in the same straight line, and AB=BC=CD=DE; show that

$$AF^2 - EF^2 = 2\,(BF^2 - DF^2).$$

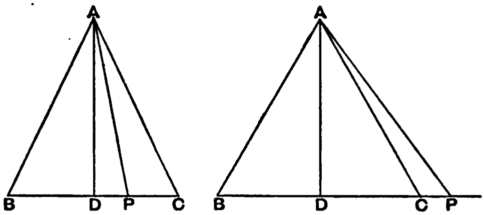
§ 34. (*Bookwork*, EUCLID, II. 1-8.)

DEFINITION.—*If a point be taken in a straight line produced, its distances from the extremities of the line are called* **Segments** *of the line, and the point is called a* **Point of External Section.**

A straight line is equal to the *sum* of the segments into which it is divided by an *internal* point, and to the *difference* between the segments into which it is divided by an *external* point.

1. If, from the vertex of an isosceles triangle, a straight line be drawn to meet the base, or base produced, the difference of the squares on this line, and on one of the equal sides of the triangle, is equal to the rectangle contained by the segments of the base.

(*A Standard Theorem.*)



In the $\triangle$ ABC, let $AB = AC$, and P be any point in BC, or BC produced, it is required to prove that

$$AB^2 \sim AP^2 = BP \cdot PC.$$

Bisect BC in D, and join AD.

$AD \perp BC,$ [§ 3, Ex. 1.

$\therefore AB^2 \sim AP^2 = BD^2 \sim DP^2;$ [§ 20, Ex. 5.

$= (BD + DP) \cdot (BD \sim DP);$ [§ 33, Ex. 1.

$= (BD + DP) \cdot (DC \sim DP);$ [Construction.

$= BP \cdot PC.$

In figure 1, when P is in BC, this result becomes

$$AB^2 - AP^2 = BP \cdot PC;$$

but in figure 2, where P is in BC produced, it is

$$AP^2 - AB^2 = BP \cdot CP.$$

Note.—It is to be observed that these two results are algebraically identical, if we assume that $CP = -PC$, and therefore $BP \cdot CP = -BP \cdot PC$; or, in other words, that the line PC changes its *sign* when P passes to the other side of C, so that the line is measured in the opposite direction.

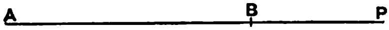
A similar use of the sign — to indicate the opposite direction is made in Trigonometry.

2. If a straight line be divided externally, the square on the straight line is equal to the sum of the squares on the segments diminished by twice the rectangle contained by the segments.

(*A Standard Theorem.*)

Use Euc. II. 7 to show that

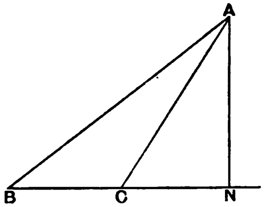
$$AB^2 = AP^2 + BP^2 - 2AP \cdot BP.$$



The above theorem may be regarded as an alternative method of stating Euc. II. 7. It shows the relation which that proposition bears to Euc. II. 4.

Note.—Here, as in Ex. 1, we observe that when the point of section is external instead of internal, the same theorem holds good if we consider that one of the segments, and therefore the rectangle contained by the segments has *changed sign*, and must be subtracted instead of added.

3. In any triangle, in which the altitude from the vertex falls beyond the base, the square on the base, together with twice the square on the altitude, is equal to the sum of the squares on the sides diminished by twice the rectangle contained by the segments of the base.



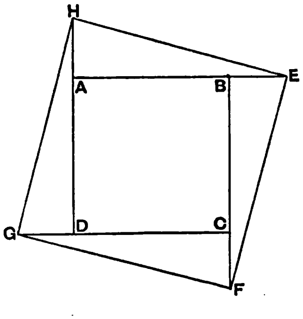
Use Euc. I. 47 and II. 7 (or Ex. 1) to show that

$$BC^2+2AN^2=AB^2+AC^2-2BN \cdot CN.$$

Compare § 32, Ex. 3, and observe that the results are algebraically identical if we assume that $CN=-NC$.

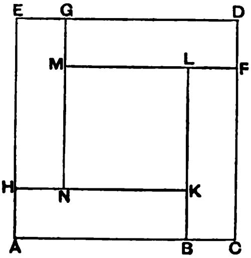
4. If, in the produced sides of a given square, four points be taken at equal distances from the angular points, towards the same parts, the figure contained by the straight lines which join these points shall be a square.

Hence obtain a second proof of Euc. II. 7.



Compare § 32, Ex. 7.

5. The square on the sum of two straight lines exceeds the square on their difference by four times the rectangle contained by the two lines.



Let AB, BC be the given lines. Show by a geometrical construction, as in the figure, that

$$(AB+BC)^2=(AB-BC)^2+4AB\cdot BC.$$

Or deduce from Euc. II. 8.

6. The square on the sum of two straight lines, together with the square on their difference, is equal to twice the sum of the squares on the two straight lines.

Use Euc. II. 4 to show that

$$(AB+CD)^2=AB^2+CD^2+2AB\cdot CD,$$

and Ex. 2 or Euc. II. 7 to show that

$$(AB-CD)^2=AB^2+CD^2-2AB\cdot CD,$$

and add the results.

For a geometrical proof see § 35, Ex. 2.

7. If the straight line AB be bisected in C and produced to D, so that the square on BD is equal to twice the square on BC, show that the rectangle AD·BD shall be equal to the rectangle AB·CD.

8. AB is divided in P and Q, so that AP = QB; show that

$$AB^2+PQ^2=2AP^2+2PB^2.$$

9. ABCD is a quadrilateral, right-angled at A and C. From B and D, BM and DN are drawn ⊥ AC. Prove AM = CN.

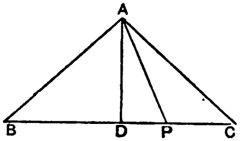
Use Euc. I. 47, § 20, Ex. 5, and § 33, Ex. 1.

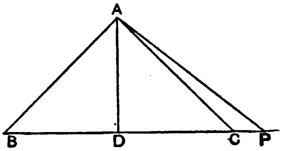
10. A, B, C, D are four points in order in the same straight line, and AB = 2BD; show that $AC^2+2CD^2=AB^2+2BD^2+3BC^2$.

§ 35. (*Bookwork*, Euclid II. 1-10.)

1. ABC is an isosceles right-angled triangle, and P is any point in the hypotenuse BC, or in BC produced; show that

$$BP^2+PC^2=2\,AP^2.$$





Bisect BC in D. Join AD, AP.

D is the mid-point of the hypotenuse of the right-angled $\triangle$ ABC.

$\therefore AD=BD=CD.$ [§ 12. Ex. 1.

and since the $\triangle$ ABC is isosceles, the median $AD \perp BC$.

[§ 3, Ex. 1.

In either figure,

$BP^2+PC^2=2BD^2+2DP^2.$ [Euc. II. 9 and 10.

$=2AD^2+2DP^2$ [Proved above.

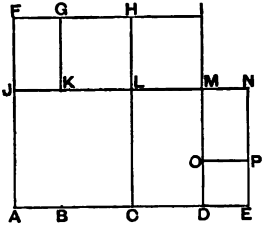
$=2AP^2.$ [Euc. I. 47.

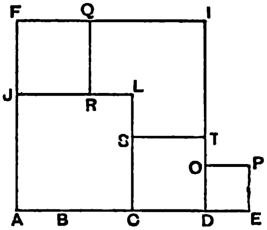
2. The square on the sum of two straight lines, together with the square on their difference, is equal to twice the sum of the squares on the two straight lines. *(A Standard Theorem.)*

Show that Euclid II. 9 and 10 are particular cases of this general theorem, and prove it by a method which shows how the one set of squares can be fitted together so as to form the other set.

Let AC, CD be the two straight lines, produce CD to E so that CE=AC, and cut off AB=DE, then we have to show that $(AC+CD)^2+(AC-CD)^2=2\ AC^2+2\ CD^2$;
that is, that $AD^2+DE^2=2\ AC^2+2\ CD^2$.

Since AB=DE, this corresponds with II. 9, if we consider AE as the line bisected in C and divided in D, while it corresponds with II. 10, if we consider BD as the line bisected in C and divided externally in A.





In figure 1, describe squares on AD, DE, AC, CE, KL, and prove FK=MP, and hence

$$AI+DP=AL+CN+KH+LI;\ \text{etc.}$$

Otherwise—

In figure 2, describe squares on AD, DE, AC, CD, FJ, and prove RL=LS=DE, and hence that

$$\text{the gnomon } QT+DP=AL,\ \text{etc.}$$

This latter construction is due to Miss Hilda Hudson. See *Nature*, December 24, 1891.

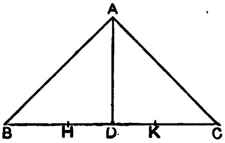
Observe, also, that since

$$AC=\tfrac{1}{2}(AD+DE),\ CD=\tfrac{1}{2}(AD-DE),$$

the theorem may also be enunciated thus :—

The sum of the squares on two straight lines is equal to twice the square on half the sum of the straight lines, together with twice the square on half their difference.

3. Points H and K are taken in the base BC of an isosceles right-angled △ ABC, such that $BK=CH=AB$; show that



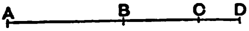
(1) $HK^2=BH^2+KC^2$;

(2) $BC^2=BH^2+KC^2+2BC\cdot HK$.

(1) Use Euc. II. 9, and show that $BH^2=2HD^2$, etc.

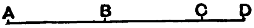
(2) Use Euc. II. 4, etc.

4. A, B, C, D are four points in order in the same straight line, and $AB=BD$; show that $AC^2+CD^2=4\,BC^2+2AC\cdot CD$.



Use Euc. II. 9 and II. 5.

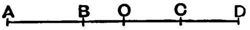
5. A, B, C, D are four points in order in the same straight line, and $AB=BC$; show that $AD^2+CD^2=4\,BD^2-2AD\cdot CD$.



Use Euc. II. 10 and II. 6.

Compare with preceding Ex. in view of the notes to § 34, Exx. 1, 2.

6. A, B, C, D are four points in order in the same straight line, and $AB>CD$; show that $AC^2+CD^2>AB^2+BD^2$.



Bisect AD in O, and use Euc. II. 9.

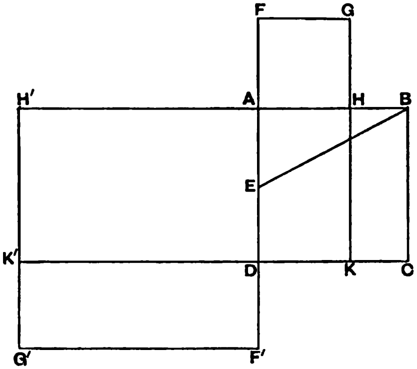
7. ABCD is a square. If E be a point in AC, or AC produced, show that the triangle whose sides are AE, CE, and the diagonal of the square on BE is right-angled.

Prove that $AE^2+EC^2=2BE^2$.

8. ABC is an isosceles triangle, and P is any point in the base BC, or BC produced; show that $BP^2+CP^2+2AB^2-2AP^2=BC^2$.

§ 36. (*Bookwork*, EUCLID II., 1-11.)

1. If, in the figure of Euclid II. 11, AD be produced to F′, so that EF′=EB, and BA be produced to H′ so that H′A=AF′, then AH′²=AB·H′B.



Complete the square AH′G′F′, and produce CD to meet H′G′ in K′.

$AF' \cdot DF' + AE^2 = EF'^2$; [Euc. II. 6.

$= EB^2$; [Hypothesis.

$= AE^2 + AB^2$; [Euc. I. 47.

$\therefore AF' \cdot DF' = AB^2$; [Subtracting equals.

or $DF'G'K' = ABCD$.

$\therefore AF'G'H' = BCK'H'$;

or $AH'^2 = BC \cdot H'B = AB \cdot H'B$.

DEFINITION.—*When a straight line is divided, either internally or externally, into two segments, such that the rectangle contained by the whole line and one segment is equal to the square on the other segment, it is said to be divided in* **Medial Section.**

In the above figure $AH^2 = AB \cdot HB$, [Euc. II. 11.

and $AH'^2 = AB \cdot H'B$. [Proved above.

Thus AB is divided internally at H and externally at H′ in medial section.

2. AB is divided internally at H and externally at H′ in medial section; show that

(1.) $AB = H'A - AH$;

(2.) $AH^2 - BH^2 = AH \cdot HB$;

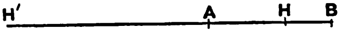
(3.) $BH'^2 - AH'^2 = H'A \cdot H'B$;

(4.) $AB^2 + BH^2 = 3AH^2$;

(5.) $AB^2 + BH'^2 = 3AH'^2$;

(6.) $AB \cdot AH = H'A \cdot HB$;

(7.) $AB \cdot H'A = AH \cdot H'B$.



(1) Show from construction of figure to Ex. 1, that

$$AB = AF' - AF, \text{ etc.}$$

(2) $AH^2 - BH^2 = AB \cdot HB - BH^2$, etc.

(4) Deduce from (2) by Euclid II. 4.

(6) Use Euc. II. 3, and II. 11.

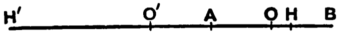
3. AB is divided internally at H and externally at H′ in medial section; O is the mid-point of AB, and O′ is the mid-point of HH′; show that

(1.) $O'A = AO$;

(2.) $AH \cdot HB = 2OH \cdot AB$;

(3.) $H'A \cdot H'B = 2H'O \cdot AB$;

(4.) $O'A^2 + O'H^2 = O'B \cdot AB$.



(1) Deduce from Ex. 2 (1).

(2) Use Ex. 2 (2), and § 33, Ex. 1.

(4) Use Euc. II. 9 and 11, and Ex. 1.

4. In the figure of Euclid II. 11, if FC meet AB in T and HK in M, and if DH meet BE in O and BF in N; show that

(1.) AK, FC, GB are parallel;

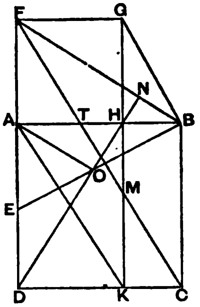
(2.) AT=HB, GH=MK, FT=MC;

(3.) DH $\perp$ BF;

(4.) AO $\perp$ DH;

(5.) $EF^2=5AE^2$;

(6.) FG, DH, CB are concurrent.



(1) Show that $\triangle$ FAK$=\triangle$CAK,
and $\triangle$ GAK$=\triangle$BAK.

(2) Use Euc. I. 34.

(3) Show that $\triangle$ FAB$\equiv\triangle$HAD, and hence that $\triangle$ HNB is equiangular with $\triangle$ HAD.

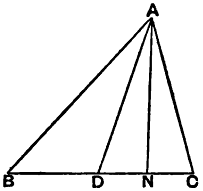
(4) Show that $\triangle$ BON is equiangular with $\triangle$ FDN, and hence that EO$=$ED, etc.

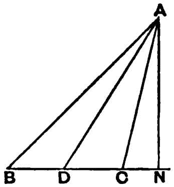
5. AB is divided in H, so that $AB \cdot HB=AH^2$, and AT is cut off equal to HB; show that $AH \cdot TH=AT^2$.

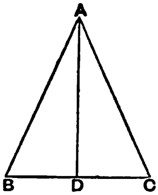
6. If L be the mid-point of AH in the above figure, show that $BL^2=5AL^2$.

§ 37. (*Bookwork*, EUCLID II. 1-13.)

1. In any triangle, the sum of the squares on two sides is equal to twice the square on half the third side, together with twice the square on the median which bisects it. (*A Standard Theorem.*)







Let AD be the median from A in the $\triangle$ ABC; it is required to prove that

$$AB^2+AC^2=2BD^2+2AD^2.$$

In figures 1 and 2, draw $AN \perp BC$, or BC produced, and of the two $\angle$s ADB, ADC, let ADB be obtuse.

$$AB^2=BD^2+AD^2+2BD \cdot DN. \qquad \text{[Euc. II. 12.}$$

$$AC^2=DC^2+AD^2-2DC \cdot DN. \qquad \text{[Euc. II. 13.}$$

But $BD=DC$;

$\therefore BD^2=DC^2$;

and $BD \cdot DN=DC \cdot DN$;

$$\therefore AB^2+AC^2=2BD^2+2AD^2.$$

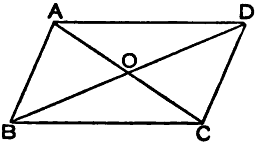
If, as in figure 3, AN coincide with AD;

$$AB^2=BD^2+AD^2, \qquad \text{[Euc. I. 47.}$$

$$AC^2=CD^2+AD^2;$$

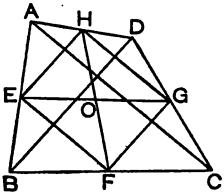
$$\therefore AB^2+AC^2=2BD^2+2AD^2.$$

2. In any parallelogram, the sum of the squares on the four sides is equal to the sum of the squares on the diagonals.



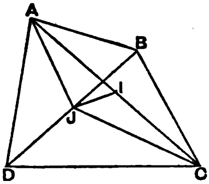
Apply Ex. 1 to $\triangle$s ABC, ACD, using § 15, Ex. 1, and § 20, Ex. 3.

3. In any quadrilateral, the sum of the squares on the diagonals is equal to twice the sum of the squares on the straight lines joining the mid-points of opposite sides.



Use § 16, Exx. 1 and 3, to show that $AC^2+BD^2=2\,(EF^2+FG^2+GH^2+HE^2)$, and apply the previous Ex.

4. In any quadrilateral, the sum of the squares on the sides is equal to the sum of the squares on the diagonals, together with four times the square on the straight line joining the mid-points of the diagonals.



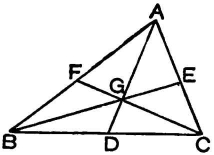
Join AJ, JC.

Apply Ex. 1 to each of the $\triangle$s ABD, CBD, AJC.

Observe that Ex. 2 is a particular case of Ex. 4; for, when ABCD is a parallelogram, I and J coincide with O.

5. In any triangle, three times the sum of the squares on the sides is equal to four times the sum of the squares on the medians.

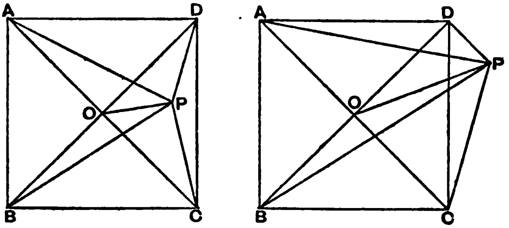
Apply Ex. 1 to each pair of sides, and add results.



6. If G be the point of intersection of the medians of a $\triangle$ ABC, show that $3(AG^2+BG^2+CG^2)=BC^2+CA^2+AB^2$.

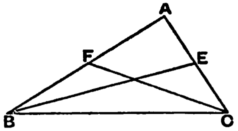
Apply Ex. 1 to each of the $\triangle$s BGC, CGA, AGB and add the results, remembering that $AG^2=4DG^2$, etc.

7. O is the centre of a square ABCD, and P is any other point; show that $PA^2+PB^2+PC^2+PD^2=2AB^2+4PO^2$.



Apply Ex. 1 to the △s PAC, PBD.

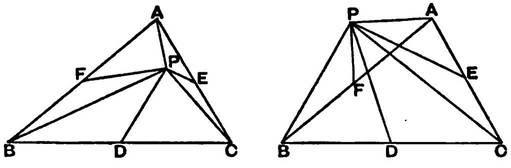
8. ABC is a triangle, right-angled at A; E and F are the mid-points of CA and AB; show that $4(BE^2+CF^2)=5BC^2$.



Apply Ex. 1. to the
△s ABC, BCA;
or use Euc. I. 47.

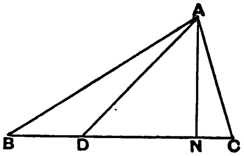
9. D, E, F are the mid-points of the sides of the △ ABC, and P is any point; show that

$$4(PA^2+PB^2+PC^2)=4(PD^2+PE^2+PF^2)+BC^2+CA^2+AB^2.$$



Apply Ex. 1 to the △s PBC, PCA, PAB.

10. D is a point in BC, the base of a △ ABC, such that 2BD=DC; show that $2AB^2+AC^2=6BD^2+3AD^2$.



Draw AN ⊥ BC.

$2AB^2=2BD^2+2AD^2+4BD \cdot DN.$

[Euc. II. 12.

$AC^2=DC^2+AD^2-2DC \cdot DN$

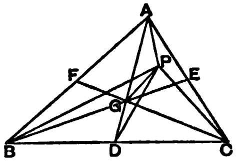
[Euc. II. 13.

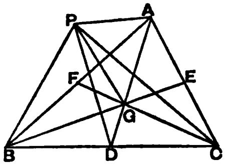
Consider the case in which ∠ ADB is acute.

11. G is the point of intersection of the medians of the △ ABC, and P is any other point; show that

$$PA^2+PB^2+PC^2=GA^2+GB^2+GC^2+3PG^2.$$

(*A Standard Theorem.*)





Apply Ex. 1 to △ PBC, and Ex. 10 to △ PAD, and deduce that $PA^2+PB^2+PC^2=2BD^2+6DG^2+3PG^2$.

Use Ex. 1 to show that

$$GA^2+GB^2+GC^2=2BD^2+6DG^2.$$

12. The hypotenuse BC of a right-angled △ ABC is trisected in D and E; show that $AD^2+AE^2=5DE^2$.

13. If two medians of a triangle be equal, the triangle shall be isosceles.

Use Ex. 1.

14. ABCD is a trapezium in which AD is parallel to BC; G, H are the mid-points of AB, CD; show that

$$AC^2+AD^2+BC^2+BD^2=AB^2+CD^2+4GH^2.$$

15. K is the point of intersection of the diagonals AC and BD of the quadrilateral ABCD, and O is the mid-point of the straight line joining the mid-points of the diagonals; show that

$$KA^2+KB^2+KC^2+KD^2=OA^2+OB^2+OC^2+OD^2+4KO^2.$$

Use Euc. II. 9 and Ex. 1.

16. ABCD is a diameter of two concentric circles, and P, Q are points on the inner and outer circles respectively; show that $PA^2+PD^2=QB^2+QC^2$.

17. ABC is a triangle, and D is a point in BC, such that $3BD=DC$; show that $3AB^2+AC^2=12BD^2+4AD^2$.

Compare Ex. 10.

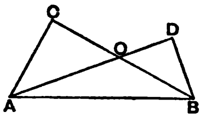
18. A, B, C, D are four points, E is the mid-point of AB; EC is divided in F, so that $2EF=FC$; FD is divided in G, so that $3FG=GD$, and P is any point; show that

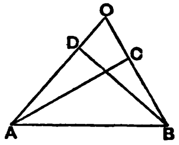
$$PA^2+PB^2+PC^2+PD^2=GA^2+GB^2+GC^2+GD^2+4PG^2.$$

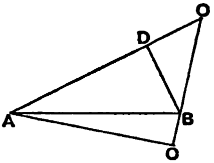
§ 38. (*Bookwork*, EUCLID, II. 1-14.)

1. Two right-angled △s ACB, ADB have the same hypotenuse AB; AD, BC intersect in O; it is required to prove that

$$AO \cdot OD = BO \cdot OC.$$







In figure 1, $\angle AOB$ is obtuse. [Euc. I. 16.

$\therefore AB^2 = AO^2 + OB^2 + 2AO \cdot OD$. [Euc. II. 12.

Similarly, $AB^2 = AO^2 + OB^2 + 2BO \cdot OC$;

$\therefore AO \cdot OD = BO \cdot OC$.

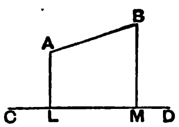
In figures 2 and 3, $\angle AOB$ is acute. [Euc. I. 16.

$\therefore AB^2 = AO^2 + OB^2 - 2AO \cdot DO$. [Euc. II. 13.

Similarly, $AB^2 = AO^2 + AO^2 - 2BO \cdot CO$;

$\therefore AO \cdot DO = BO \cdot CO$.

DEFINITION.—*The* **Orthogonal Projection** *of one straight line on another is the part of the latter intercepted between perpendiculars let fall on it from the extremities of the former.*



Thus, if, from A and B, AL, BM be drawn perpendicular to CD, LM is the projection of AB on CD. It is clear that the length LM depends only on the length of AB and the angle which AB makes with CD.

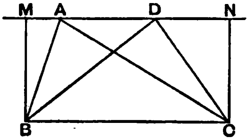
In Ex. 1, CO is the projection of AO on BO, and DO is the projection of BO on AO. The theorem may, therefore, be stated thus:—

If there be two straight lines, the rectangle contained by the first and the projection on it of the second is equal to the rectangle contained by the second and the projection on it of the first.

(*A Standard Theorem.*)

2. In a trapezium the sum of the squares on the diagonals is equal to the sum of the squares on the oblique sides, together with twice the rectangle contained by the parallel sides.

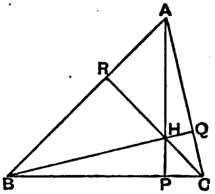
Draw BM, CN $\perp$ AD, or AD produced. Apply Euc. II. 12 or 13 to $\triangle$s ABD, ACD. Then use Euc. II. 1.



3. H is the point of intersection of the altitudes AP, BQ, CR of the acute-angled $\triangle$ ABC; show that

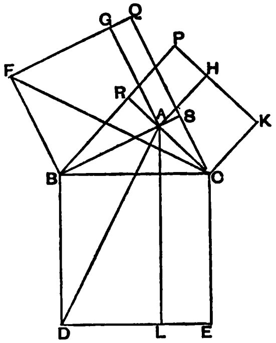
$$BC^2+CA^2+AB^2=2(AP\cdot AH+BQ\cdot BH+CR\cdot CH).$$

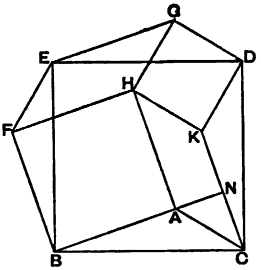
Apply Euc. II. 13 to $\triangle$s BHC, CHA, AHB, and add the results.



Examine the case in which the triangle is obtuse-angled.

4. Prove, by means of a geometrical construction, that in obtuse-angled triangles the square on the side opposite the obtuse angle is equal to the sum of the squares on the other two sides, together with twice the rectangle contained by either of these sides, and the projection on it of the other side (Euc. II, 12).





In figure 1, construct squares as in Euclid I. 47, and draw BRP, CSQ $\perp$ AC, AB.

As in Euc. I. 47, show that

$$BL = FS = AF + AQ = AB^2 + BA \cdot AS\,;$$

and similarly, that $$CL = CA^2 + CA \cdot AR.$$

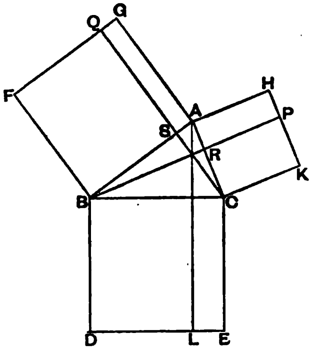
Join, BH, CG, and use Euc. I. 4 and I. 39 to prove $AQ = AP$.

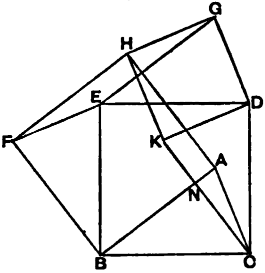
Otherwise—

In figure 2, on BC describe the square BCDE, and on BE, ED, CD describe the $\triangle$s BFE, EGD, CKD congruent with $\triangle$BAC. Complete the square ABFH, and join HG, HK. Show that the square BCDE = the figure BACKDGEF, and show that $AF = AB^2$, $DH = AC^2$, and that each of the parallelograms AK and EH is equal to the rectangle $BA \cdot AN$.

Compare § 20, Ex. 8.

5. Prove, by means of a geometrical construction, that in any triangle the square on the side opposite an acute angle is equal to the sum of the squares on the other two sides, diminished by twice the rectangle contained by either of these sides, and the projection on it of the other side (Euc. II. 13).





The constructions are similar to those of the previous Ex.

In figure 1, show that $BL = FS = AB^2 - AB \cdot AS$, etc.

Otherwise—

In figure 2, $AHFBC + GHKCD = GHFBCD + AHKC$.

Hence, show that $AB^2 + AC^2 + 2\triangle ABC = BC^2 + EFHG + AHKC + 2\triangle ABC$.

Show, also, that $EFHG = AHKC = AB \cdot AN$, and deduce the theorem.

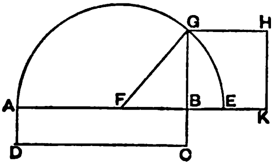
6. If a rectangle and a square be equal in area, the square shall have the lesser perimeter.

Use Euc. II. 14 to show that

$$ABCD = BKHG,$$

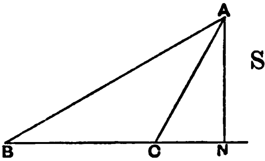
and show that

perimeter of $AC = 4FG$,
perimeter of $BH = 4BG$, etc.



Compare § 33, Ex. 8.

7. The ∠ ACB of a △ ABC is four-thirds of a right angle; show that $AB^2=AC^2+BC^2+AC\ BC$.



Show that AC=2CN, [§ 13, Ex. 1.
and use Euc. II. 12.

8. The ∠ACB of a △ ABC is two-thirds of a right angle; show that $AB^2=AC^2+BC^2-AC{\cdot}BC$.

Use § 13, Ex. 1, and Euc. II. 13.

9. If, in the △ ABC, AC=BC, and ∠ ACB=four-thirds of a right angle; show that $AB^2=3AC^2$.

10. O is a point within the △ ABC such that

∠ BOC=∠COA=∠AOB; show that

$$BC^2+CA^2+AB^2=2(OA^2+OB^2+OC^2)+OB{\cdot}OC+OC{\cdot}OA+OA{\cdot}OB.$$

11. If, in the △ ABC, BQ and CR are the altitudes from B and C, and the ∠s B, C are both acute; show that

$$BC^2=BA{\cdot}BR+CA{\cdot}CQ.$$

Use Euc. II. 12 or 13, and Ex. (1).

Consider the case in which the ∠ B is obtuse.

12. H is the orthocentre of the acute-angled △ ABC and AP, BQ, CR are the altitudes; show that

(1) $AH{\cdot}HP=BH{\cdot}HQ=CH{\cdot}HR$.

(2) $AH^2+BC^2=BH^2+CA^2=CH^2+AB^2$.

For (1) use Ex. 1, for (2) use § 20, Ex. 5 or Euc. II. 12.

13. In the △ ABC, AC=BC, and AN ⊥ BC, or BC produced; show that $AB^2=2BC{\cdot}BN$.

Projections.

14. If two straight lines be equal and parallel, their projections on any other straight line shall be equal.

Enunciate and prove two converses to this theorem.

15. A, B are points on the bisector of the ∠ XOY; P, Q are their projections on OX; and R, S are their projections on OY; show that PR ∥ QS.

Note.—The projection of a point on a straight line is the foot of the perpendicular drawn from the point to the line.

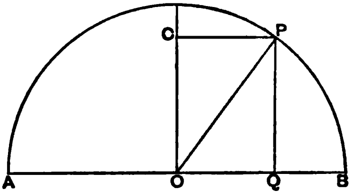
CHAPTER II.

PROBLEMS.

§ 39. *Problems which follow directly from known propositions.*

1. Divide a given straight line into two parts so that the rectangle contained by the parts shall be equal to a given square.

This is a converse problem to Euc. II. 14, and the following construction is at once suggested.



Construction—

Bisect AB in O, and describe a circle with O as centre and OA as radius.

Through O draw $OC \perp AB$ and equal to a side of the given square.

Through C draw $CP \parallel AB$, meeting the circle in P, and through P draw $PQ \perp AB$.

Q shall be the required point of section.

Proof—

$$AQ \cdot QB + OQ^2 = AO^2; \qquad \text{[Euc. II. 5.}$$

$$= PO^2;$$

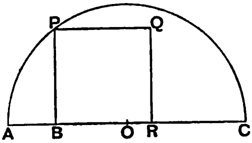
$$= PQ^2 + OQ^2; \qquad \text{[Euc. I. 47.}$$

$$\therefore AQ \cdot QB = PQ^2.$$

Observe that if PC be produced to meet the circle again in P′, and P′Q′ be drawn ⊥ AB, a second point will be found to satisfy the condition, but that, if OC > OA, no point satisfies the condition.

Observe, also, that APB is a right-angled triangle, and compare § 32, Ex. 1.

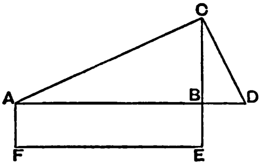
2. Construct a square equal to *n* times a given square.



Let AB be a side of the given square.
Produce AB to C, so that
$BC = nAB$,
and proceed as in Euc. II. 14.

This problem may also be solved by means of Euc. I. 47.

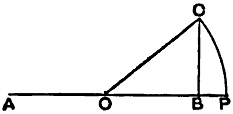
3. Construct a rectangle equal to a given square and having one side equal to a given straight line.



Let BC be a side of the given square and be ⊥ AB the given straight line.

Produce AB and draw CD⊥AC, etc.
Use § 32, Ex. 1.
Compare Euc. II. 14.

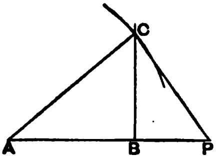
4. Produce a given straight line, so that the rectangle contained by the whole line thus produced, and the part produced, shall be equal to a given square.



Let AB be the given straight line, CB a side of the given square.

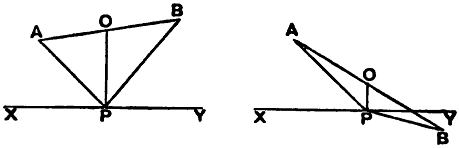
Produce AB and make OP=OC where O is mid-point of AB.
Use Euc. II. 6.
Compare Euc. II. 14.

5. Produce a given straight line AB to P, so that AB·AP shall be equal to the square on a given straight line AC.



See § 32, Ex. 1 (2).

6. A, B are fixed points and XY is a given straight line; find a point P in XY, such that AP^2+BP^2 may be a minimum.



Bisect AB in O, and draw OP⊥XY.

Use § 37, Ex. 1, and § 8, Ex. 1.

7. Construct a rectangle having given the area and the perimeter.

Observe that in Ex. 1, AB is one-half the perimeter.

8. Construct a square which shall be equal to
(1) A given parallelogram.
(2) A given triangle.

9. Construct a rectangle equal to the sum of two given squares, so that one side of the rectangle may be equal to a given straight line.

10. Construct a rectangle equal to the difference of two given squares, so that one side of the rectangle may be equal to a given straight line.

11. Divide AB in P, so that
(1) $AB^2+PB^2=3AP^2$.
(2) $AP^2-PB^2=AP\cdot PB$.

See § 36, Ex. 2. Can P be a point of external section?

12. Produce AB to P so that AB·BP may be equal to a given square.

See § 32, Ex. 1.

13. Divide a straight line so that the rectangle contained by the segments may be a maximum.

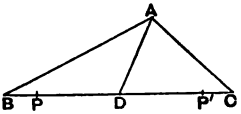
Use Euc. II. 5.

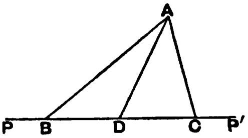
14. Divide a straight line so that the sum of the squares on the segments may be a minimum.

Use Euc. II. 9.

§ 40. *Analysis and Synthesis.*

1. Divide one side of a triangle, so that the sum of the squares on its segments may be equal to the sum of the squares on the other two sides of the triangle.





Analysis.

Suppose that BC *is divided in* P, *so that*

$$AB^2+AC^2=PB^2+PC^2.$$

If D *be the mid-point of* BC,

$$AB^2+AC^2=2AD^2+2BD^2.$$ [§ 37, Ex. 1.

Also $$PB^2+PC^2=2PD^2+2BD^2.$$ [Euc. II. 9, 10.

Hence we see that $PD=AD$.

This suggests the following Construction—

Bisect BC in D.
Join AD and cut off $DP=DA$.
P is the point required.

Proof—

$$AB^2+AC^2=2AD^2+2BD^2.$$ [§ 37, Ex. 1.

$$=2PD^2+2BD^2.$$ [Construction.

$$=PB^2+PC^2.$$ [Euc. II. 9, 10.

A second solution can be obtained by cutting off

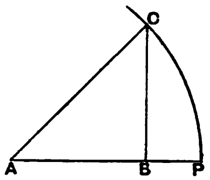
$$DP'=AD.$$

Observe that P is in BC, or BC produced, according as $\angle A$ is obtuse or acute, and falls on B when $\angle A$ is right. Compare § 12, Ex. 6.

2. Produce a given straight line AB to P, so that

$$AB^2+BP^2=2AP\cdot BP.$$

Suppose P *found. Then*



$$AB^2+BP^2=2AP\cdot BP,$$

$$\therefore\ 2AB^2+BP^2=AB^2+2AP\cdot BP$$

$$=AP^2+BP^2.\ \text{[Euc. II. 7.}$$

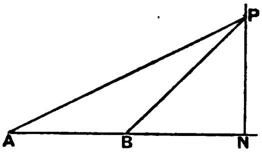
$$\therefore\ AP^2=2AB^2,$$

from which the Construction easily follows.

3. On a given base AB, construct a △ PAB such that

$$PA^2=PB^2+3AB^2.$$

Suppose PAB *the required triangle, and draw* $PN \perp AB$.



$$PA^2=AB^2+PB^2+2AB\cdot BN,\ \text{[Euc. II. 12.}$$

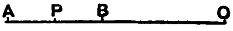
but $PA^2=AB^2+PB^2+2AB\cdot AB$, [Hypothesis.

$$\therefore\ BN=AB.$$

Hence we obtain a construction for PN the *locus* of P.

4. A, B, C are three points in order in the same straight line. Divide AB in P so that $AP\cdot AC=2PB^2+PB\ BC$.

Suppose P *found, then*



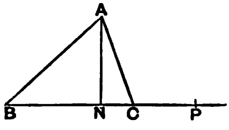
$$AP\cdot(AP+PB+BC)=$$

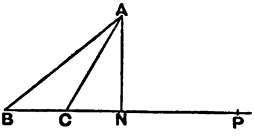
$$PB\cdot(PB+PB+BC),$$

which is evidently satisfied when

$$AP=PB.$$

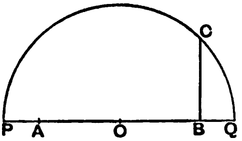
5. ABC is a triangle in which AB>AC; produce BC to P, so that $BC\cdot CP=AB^2-AC^2$.





Suppose P found. Draw $AN \perp BC$, and use § 20, Ex. 5, and § 33, Ex. 1, to show that $NP=BN$.

6. Construct a rectangle, having given the area and the difference of two adjacent sides.



Let AB be the given difference, BC^2 the given area.

Produce AB to P and Q, and suppose $PB \cdot BQ = BC^2$.

Then $PB - BQ = AB$.

$\therefore PA = BQ$.

Also $PB \cdot BQ = OQ^2 - OB^2$, if O be the mid-point of AB,

and $BC^2 = OC^2 - OB^2$. $\therefore OQ = OC$.

7. Produce AB to P, so that $AB^2 + AP^2 = 2AP \cdot BP$.

Compare Ex. 2.

8. Divide a given straight line externally, so that twice the rectangle contained by the segments may be equal to the difference of their squares.

Here $AP^2 = BP^2 + 2AP \cdot BP$.

Write $AB + BP$ for AP and use II. 4, etc.

9. A, B, C are three points in a straight line; find a point P in AC produced such that

$$BP^2 = AP \cdot BC.$$

It will be found that the required condition reduces to

$$BP \cdot CP = AB \cdot BC.$$

If by Euc. II. 14 a square be formed equal to $AB \cdot BC$, we have the case of § 39, Ex. 3.

10. O is the mid-point of AB; divide AB in P so that

$$4AP \cdot PB = 3AO^2.$$

11. Divide AB in P, so that $AB^2 + PB^2 = 2AP^2$.

Use Euc. II. 7, etc., to show that $(AB + AP)^2 = 3AB^2$. Then use § 39, Ex. 4.

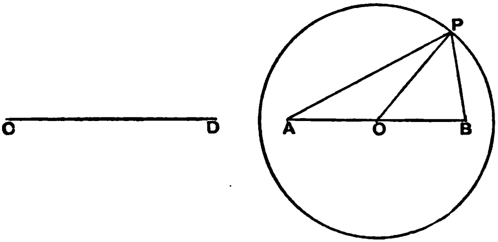
12. P is a point in AB; find a point Q in AB produced, so that $AQ^2 + BQ^2 = 2AP^2 + 2PB^2$.

13. Produce AB both ways to P and Q, so that $PA \cdot AQ$ and $PB \cdot BQ$ may be respectively equal to two given squares.

§ 41. *Loci and intersection of Loci.*

1. Find the locus of a point, the sum of the squares on whose distances from two given points is equal to a given square.

(A Standard Locus.)



Let P be any point which satisfies the condition

$$PA^2+PB^2=CD^2.$$

Bisect AB in O. Join PA, PB, PO.

$$PA^2+PB^2=2AO^2+2PO^2, \qquad [\S\ 37, \text{Ex. } 1.$$

$$\therefore\ 2AO^2+2PO^2=CD^2;$$

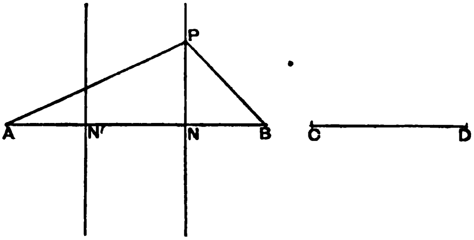
$$\therefore\ PO^2=\tfrac{1}{2}CD^2-AO^2.$$

Thus PO is of constant length, and P must lie in the circumference of the circle whose centre is O and radius PO, and any point in this circumference satisfies the condition.

When $\frac{1}{2}CD^2>AO^2$, a line equal to PO may be constructed by using § 20, Ex. 2, and § 22, Ex. 7. When $\frac{1}{2}CD^2=AO^2$, O is the only point which satisfies the condition.

When $\frac{1}{2}CD^2<AO^2$, there is no locus.

2. Find the locus of a point, the difference between the squares on whose distances from two given points is equal to a given square. *(A Standard Locus.)*



Let $PA^2 \sim PB^2 = CD^2$.

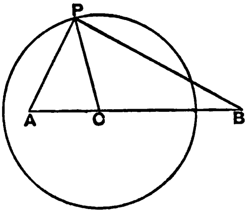
Draw $PN \perp AB$.

Then $PA^2 \sim PB^2 = AN^2 \sim NB^2$.

Thus P must be a point in a straight line drawn ⊥ AB through a point N in AB, such that $AN^2 \sim NB^2 = CD^2$.

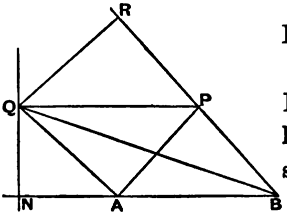
The point N may be found as in § 27, Ex. 9. There will evidently be two such points.

3. A and B are two fixed points; find the locus of P, when $2PA^2 + PB^2$ is constant.



Divide AB in C so that $CB = 2AC$, and use § 37, Ex. 10.

4. The base AB and the sum of the sides PA, PB of a △ PAB are given; find the locus of the point in which the bisector of the exterior angle at P meets the straight line through A perpendicular to PA.

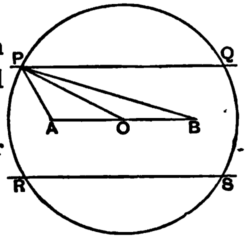


Let Q be one position of the point.

Draw $QR \perp PB$, $QN \perp AB$.

Prove $QR = QA$ and $PR = PA$; hence, show that $QB^2 - QA^2$ is constant, and use Ex. 2.

5. Construct a triangle, having given the base, the area, and the sum of the squares on the two sides.



The vertex of the triangle must lie on the locus found in § 23, Ex. 5, and also on the locus found in Ex. 1.

It is, therefore, one of the four points of intersection.

6. Find the locus of the vertex of a △ ABC if the sum of the squares on the medians BE, CF is constant.

In figure to § 37, Ex. 5, show that BG^2+CG^2 is constant.

Use Ex. 1 to show that DG and hence DA is constant.

7. A, B are two fixed points; find the locus of P, when $3PA^2+PB^2$ is constant.

See § 37, Ex. 17.

8. A, B, C are fixed points; find the locus of P, when $PA^2+PB^2+PC^2$ is constant.

Use § 37, Ex. 11. Compare Exx. 1, 3.

9. The base AB and the difference of the sides PA, PB of a △ PAB are given, find the locus of the point in which the bisector of the ∠P meets the straight line through A perpendicular to PA.

Compare Ex. 4 and use Ex. 2.

10. Construct a triangle, having given the base, the area, and the difference of the squares on the two sides.

Use Ex. 2 and § 23, Ex. 5.

11. Construct a triangle, having given the base, and the sum and difference of the squares on the two sides.

12. Construct a triangle, having given the base, an angle adjacent to the base, and the sum of the squares on the two sides.

13. Construct a triangle, having given the base, an angle adjacent to the base, and the difference of the squares on the two sides.

14. Construct a right-angled triangle, having given the hypotenuse, and the difference of the squares on the two sides.

APPENDIX I.

Enunciations of the Propositions of

EUCLID, BOOK II.

1. If there be two straight lines, one of which is divided into any number of parts, the rectangle contained by the two lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.

2. If a straight line be divided into any two parts, the square on the whole line shall be equal to the sum of the rectangles contained by the whole line and each of the parts.

3. If a straight line be divided into any two parts, the rectangle contained by the whole line and one of the parts shall be equal to the square on that part, together with the rectangle contained by the parts.

4. If a straight line be divided into any two parts, the square on the whole line shall be equal to the sum of the squares on the parts, together with twice the rectangle contained by the parts.

5. If a straight line be divided into two equal parts, and also into two unequal parts, the rectangle contained by the unequal parts, together with the square on the line between the points of section, shall be equal to the square on half the line.

6. If a straight line be bisected and produced to any point, the rectangle contained by the whole line thus produced and the part of it produced, together with the square on half the line bisected, shall be equal to the square on the line made up of the half and the part produced.

7. If a straight line be divided into any two parts, the square on the whole line, together with the square on one of the parts, shall be equal to twice the rectangle contained by the whole line and that part, together with the square on the other part.

8. If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, shall be equal to the square on the line made up of the whole and the first part.

9. If a straight line be divided into two equal parts, and also into two unequal parts, the sum of the squares on the two unequal parts shall be double the sum of the squares on half the line and on the line between the points of section.

10. If a straight line be bisected and produced to any point, the sum of the squares on the whole line thus produced, and on the part of it produced, shall be double the sum of the squares on half the line bisected, and on the line made up of the half and the part produced.

11. To divide a straight line into two parts so that the rectangle contained by the whole line and one part shall be equal to the square on the other part.

12. In an obtuse-angled triangle, the square on the side subtending the obtuse angle is greater than the sum of the squares on the sides containing the obtuse angle by twice the rectangle contained by either of these sides, and the produced part of it intercepted between the perpendicular let fall on it from the opposite vertex and the obtuse angle.

13. In every triangle, the square on the side subtending an acute angle is less than the sum of the squares on the sides containing that angle, by twice the rectangle contained by either of these sides and the part of it intercepted between the perpendicular let fall on it from the opposite vertex and the acute angle.

14. To describe a square that shall be equal to a given recti lineal figure.

Enunciations of Standard Theorems and Loci.

§ 31, Ex. 1. If A, B, C, D be four points in order on the same straight line, the rectangle AC·BD shall be equal to the sum of the rectangles AD·BC and AB·CD.

§ 31, Ex. 3. The square on any straight line is four times the square on half the line.

§ 32, Ex. 1. In a right-angled triangle, the square on the altitude from the right angle is equal to the rectangle contained by the segments into which the altitude divides the hypotenuse; and the square on either of the sides which contain the right angle is equal to the rectangle contained by the hypotenuse and its adjacent segment.

§ 33, Ex. 1. The rectangle contained by the sum and the difference of two straight lines is equal to the difference between their squares.

§ 33, Ex. 4. The difference between the squares on two sides of a triangle is equal to twice the rectangle contained by the base and the segment of the base intercepted between its mid-point and the foot of the altitude.

§ 34, Ex. 1. If, from the vertex of an isosceles triangle, a straight line be drawn to meet the base, or base produced, the difference of the squares on this line, and on one of the equal sides of the triangle, is equal to the rectangle contained by the segments of the base.

§ 34, Ex. 2. If a straight line be divided externally, the square on the straight line is equal to the sum of the squares of the segments diminished by twice the rectangle contained by the segments.

§ 35, Ex. 2. The square on the sum of two straight lines, together with the square on their difference, is equal to twice the sum of the squares on the two straight lines.

§ 37, Ex. 1. In any triangle, the sum of the squares on two sides is equal to twice the square on half the third side, together with twice the square on the median which bisects it.

§ 37, Ex. 11. G is the point of intersection of the medians of the △ ABC, and P is any other point; show that

$$PA^2+PB^2+PC^2=GA^2+GB^2+GC^2+3PG^2.$$

§ 38, Ex. 1. If there be two straight lines, the rectangle contained by the first and the projection on it of the second is equal to the rectangle contained by the second and the projection on it of the first.

§ 41, Ex. 1. Find the locus of a point, the sum of the squares of whose distances from two given points is equal to a given square.

§ 41, Ex. 2. Find the locus of a point, the difference of the squares of whose distances from two given points is equal to a given square.

APPENDIX II.

MISCELLANEOUS DEDUCTIONS

EUCLID, BOOKS I. AND II.

§§ 1-14. EUCLID I., 1-32.

Theorems.

1. A and B are two fixed points, M is the mid-point of AB, and P is any other point in the same straight line; show that when P is in AB,

$$PA \sim PB = 2PM,$$

and that when P is in AB produced,

$$PA + PB = 2PM.$$

2. OA and OB are two fixed straight lines, OM is the bisector of the $\angle AOB$, and OP is any other straight line through O; show that when OP falls within the $\angle AOB$,

$$\angle POA \sim \angle POB = 2\angle POM,$$

and that when OP falls without the $\angle AOB$,

$$\angle POA + \angle POB = 2\angle POM.$$

3. ABC is a triangle in which AB = AC, D is a point in the bisector of $\angle A$, DB and DC are joined and produced to E and F, so that BE = CF; prove that $\triangle$ AEF is isosceles.

4. ABC is a triangle whose base BC is trisected in D and E; if AD = AE, show that AB = AC.

5. In the quadrilateral ABCD, AB = CD and AC = BD. If AB and CD meet in O, show that OA = OD.

6. In $\triangle$ ABC, AB = AC and the straight line PQ meets AB in P, and AC produced in Q; show that $\angle APQ > \angle AQP$.

7. In any quadrilateral ABCD the sum of the exterior angles at A and C is greater than either of the interior angles at B or D.

8. ABC is a triangle in which $AB=AC$; P is a point in BC, and Q in BC produced; show that $AQ>AB$ and $AP<AB$.

9. Half the perimeter of a triangle is greater than any one side and less than any two sides.

10. D is the mid-point of the side BC of the $\triangle ABC$. If $AB>AC$, show that $\angle BAD<\angle CAD$.

11. Q and R are points in BC, and P is any point within the $\triangle ABC$; prove that $BC+CA+AB>QR+RP+PQ$.

12. ABCD is a quadrilateral whose diagonals intersect in O, and P is any point within the $\triangle ABO$; show that

$$PA+PB+PC+PD<AC+BD+2AB.$$

13. If, in the ▱ABCD, E be a point in AB, or AB produced, such that $CE=CD$, show that ED bisects the $\angle AEC$.

Examine the case in which E is a point in BA produced.

14. In the right-angled $\triangle ABC$, $AB=AC$, AC is produced to D, so that $CD=BC$, and CD to E, so that $DE=BD$; show that

$\angle BEA$ is the eighth part of a right angle.

15. In the $\triangle ABC$, D is the mid-point of BC, DO is $\perp$ BC and O is a point in DO such that $AO=BO$; show that $\angle BOC=2\angle BAC$.

16. In the $\triangle ABC$, D is a point in AC, or AC produced, such that $AD=AB$; show that $\angle CBD=\frac{1}{2}(\angle B \sim \angle C)$.

17. ABC, ABD are adjacent supplementary angles, through P a point on the bisector of $\angle ABC$, PQR is drawn $\parallel$ CD meeting AB in Q and the bisector of $\angle ABD$ in R; show that $PQ=QR$.

18. In $\triangle ABC$, $AB>AC$, BI, CI are the bisectors of $\angle$s B and C, and IX $\perp$ BC; show that $\angle BIX>\angle CIX$.

19. ABC is a triangle, right-angled at A, and $AB=AC$, from E, the mid-point of AC, EN is drawn $\perp$ BC; show that $BN=3EN$.

20. ABC is a triangle, right-angled at A, AN is drawn $\perp$ BC and produced to P, so that $AP=AB$. If $\angle ACP$ be a right angle, show that $PC=AN$.

21. The side BC of the $\triangle ABC$ is produced to D, and the bisectors of the $\angle$s ABC, ACD meet in J; show that $\angle BJC=\frac{1}{2}\angle BAC$. (*This is the principle of Hadley's Sextant.*)

22. If, in a quadrilateral ABCD, $\angle A=\angle B$ and $\angle C=\angle D$; show that AB $\parallel$ DC and $AD=BC$.

23. Prove that every quadrilateral figure, not a square or rectangle, must have at least one acute and at least one obtuse angle.

24. Show that no convex polygon can have more than three interior angles acute.

25. Show that an equilateral triangle, a regular decagon and a regular fifteen-sided figure may be placed so as to have one vertex in common, and exactly fill up the space round a point.

26. Show that the number of diagonals of a polygon of n sides is $\frac{1}{2}n(n-3)$.

27. In the $\triangle ABC$, BC is produced to X, from BX, BD is cut off equal to AB, and from CX, CE is cut off equal to AC; show that

(1) $DE=BC+CA-AB$,
(2) $\angle AED=\frac{1}{2}\angle C$,
(3) $\angle ADE=\text{rt.}\angle+\frac{1}{2}\angle B$,
(4) $\angle DAE=\frac{1}{2}\angle A$.

28. In the $\triangle ABC$, $AB>AC$, AK is the bisector of the $\angle A$, from AB, AG is cut off equal to AC, and GH parallel to AK meets BC in H, show that

(1) $CK=GK=HK$,
(2) $\angle GHK=\text{rt.}\angle-\frac{1}{2}(\angle C-\angle B)$,
(3) $\angle GKH=\angle C-\angle B$.

29. In the $\triangle ABC$, BQ and CR are the altitudes from B and C; show that the straight lines drawn at right angles to BR, RQ, QC from their mid-points are concurrent.

30. ABC is a triangle, right-angled at A, D is the mid-point of BC, and the bisector of $\angle A$ meets the perpendicular drawn to BC through D in L; show that $DL=AD$.

31. The straight line joining the right angle of a right-angled triangle to a point in the hypotenuse, other than the mid-point, is greater than one of the segments of the hypotenuse base and less than the other.

32. If, in the $\triangle ABC$, BC be produced to X and AK be the bisector of the $\angle A$; show that $\angle ABX+\angle ACX=2\angle AKX$.

33. In the $\triangle ABC$ the bisector of the $\angle$s B and C meet in I, and $PIQ \parallel BC$ meets AB and AC in P and Q, from P and Q lines are drawn $\perp$ BI and CI meeting BC in R and S; show that

$$BC=PQ+RS.$$

34. ABC is a triangle having $AB=AC$, and $AB<BC$; from BA, BC equal parts BD, BE are cut off and ED produced meets CA produced in F. Show that $3\angle ADE+\angle AFD=4$ right angles. (Compare § 13, Ex. 14.)

35. ABC is an isosceles triangle, the $\angle A$ being a right angle; D is any point in AC. From C and D, CF and DF are drawn at right angles to BC and BD respectively. Show that $DF=DB$.

§§ 15-21. Euclid I.

Theorems.

36. ABCD is a parallelogram, and equilateral △s ABE, CDF are described on AB, CD towards the same parts; show that AEFD, BEFC, are parallelograms.

37. In the ▱ ABCD, AB is produced to E and CD to F, so that AE = CF; show that BEDF is a parallelogram.

38. If, through the vertices of a parallelogram, straight lines be drawn making equal angles, towards the same parts, with the sides of the parallelogram; show that the figure contained by these four straight lines, when produced to meet, is also a parallelogram.

39. P is any point in the side CD of the ▱ ABCD. From C and D, CQ and DR are drawn parallel to PB and PA, meeting AB produced in Q and R. Show that the length of QR is independent of the position of P.

40. ABCD is a trapezium and $AD \parallel BC$; E, F, G, H are the mid-points of AB, BD, AC, CD; if AF and DG meet in a point on BC, show that $BC = 2AD$, and $EF = FG = GH$.

41. ABC, DEF are two triangles, AB, BC are equal, parallel and drawn in the same direction as DE, EF; if AD, BE, CF be joined, show that three parallelograms are formed, one of which is equal to the sum of the other two.

42. In the △ ABC, E and F are the mid-points of AC and AB, BE and CF meet in G; show that

$$\triangle ABC = 3 \triangle CGB = 4 \triangle CEF = 12 \triangle EFG.$$

43. In the △ ABC, D and E are the mid-points of BC and CA, and K is the mid-point of AD; show that

$$\triangle ABC = 8 \triangle KBE.$$

44. OAB and OCD are two straight lines which meet in O; show that $\triangle BCD > \triangle ACD$.

45. The parallelogram formed by joining the mid-points of the adjacent sides of a quadrilateral is a rhombus or a rectangle according as the quadrilateral is a rectangle or a rhombus.

46. The area of a trapezium is determined when its altitude and the sum of its parallel sides are given.

47. ABCD is a rectangle; P, Q are points in AD, BC such that 8AP=AD, 8BQ=5BC ; show that 8 area APQB=3 △ BCD.

48. In the △ ABC, D, E, F are the mid-points of BC, CA, AB; if the ▱ FBEH be completed, show that HC=AD.

49. The distances of the mid-point of the base of a triangle from the foot of the perpendicular drawn from either extremity of the base to the bisector of the vertical angle is equal to half the difference between the sides.

50. If in a trapezium one of the parallel sides be twice the other, the point of intersection of the diagonals shall be a point of trisection of each diagonal.

51. O is the point of intersection of the diagonals of the trapezium ABCD, in which AD ∥ BC. Through O, POQ is drawn ∥ AD, meeting AB and CD in P and Q; show that PO=OQ.

52. Three parallel straight lines AD, BE, CF are drawn through the vertices of the △ ABC meeting the opposite sides (produced when necessary) in D, E, F; show that △ DEF=2 △ ABC.

53. In the isosceles △ ABC, AB=AC, and points P and Q are taken in AB, AC such that AP=CQ; show that △ ACP=△ BCQ.

54. If the straight line which joins the mid-points of the diagonals of a quadrilateral be parallel to one side, it shall also be parallel to the opposite side.

55. D, E, F are the mid-points of BC, CA, AB, the sides of the △ ABC; AL, BM, CN are drawn perpendicular to EF, FD, DE; show that AL, BM, CN are concurrent.

56. A, B and C are three points in order on a straight line, such that BC=2 AB. Through these points parallel straight lines AP, BQ and CR are drawn meeting a given straight line DE in P, Q and R; show that 2 AP ± CR=3 BQ according as A and C are on the same or on opposite sides of DE.

57. ABCD is a ▱, BE parallel to AC and CE parallel to DB meet in E; AE, DE meet BD, AC in H, K; show that BC=2 HK.

58. O is any point within the △ ABC and ▱s BOCD, COAE, AOBF are completed; show that △ DEF=△ ABC.

59. OA, OB and OC are three straight lines such that ∠ BOC = ∠ COA=∠ AOB. From any point P perpendiculars are drawn to OA, OB and OC, produced if necessary; show that one of these perpendiculars is equal to the sum of the other two.

60. A, B, C are three points, AB is bisected at D and DC is divided at G, so that $GC=2\,DG$; through A, B, C, G parallel straight lines AA', BB', CC', GG', are drawn, meeting a straight line PQ in A', B', C', G'; show that $AA'\pm BB'\pm CC'=\pm 3\ GG'$, the negative signs being taken when B, C, G are on the opposite side of PQ from A.

61. A straight line XY is drawn through G, the point of intersection of the medians of the △ ABC; through A, B, C parallel straight lines AA', BB', CC' are drawn, meeting XY in A', B', C'; show that $AA'\pm BB'\pm CC'=0$, the negative signs being taken when B, C are on the opposite side of XY from A.

62. ABCD is a quadrilateral, E and F are the mid-points of AC and BD, and G is the mid-point of EF; through A, B, C, D parallels AA', BB', CC', DD' are drawn, meeting XY a straight line through G in A', B', C', D'; show that

$$AA'\pm BB'\pm CC'\pm DD'=0,$$

the negative signs being taken when B, C, D are on the opposite side of XY from A.

63. A_1, A_2, A_3 are three equidistant points on one straight line, B_1, B_2, B_3 are three equidistant points on another straight line; three parallel straight lines A_1P_1, A_2P_2, A_3P_3 meet three other parallel straight lines B_1P_1, B_2P_2, B_3P_3, in P_1, P_2, P_3; show that P_1, P_2, P_3 are collinear.

64. ABCD is a parallelogram, P is a point in AD, and PQ is drawn parallel to AB to meet BC in Q, R is a point in AB, and RS is drawn parallel to AP to meet CD in S; show that 2△AQS = ▱BD - ▱PR.

65. P is any point on the diagonal AC of the ▱ ABCD, or on AC produced; show that △ PAB = △ PAD. Conversely, if the △ PAB = △ PAD, show that P must be a point on the diagonal of the parallelogram which has AB, AD as two adjacent sides.

66. ABCD is a parallelogram and P is any point, through P are drawn GH ∥ BC and EF ∥ AB, meeting the sides of ABCD in E, F, G, H; show that EG, HF, and one of the diagonals of ABCD are concurrent.

67. ABCD is a quadrilateral; AB, CD meet at E; AD, BC meet at F; P, Q, R are the mid-points of AC, BD, EF; show that P, Q, R are collinear.

68. ABC, CDE are triangles of equal area, so placed that AC, CE and also BC, CD form straight lines; show that the straight line joining the mid-points of AD and BE passes through C.

69. The diagonals of a quadrilateral intersect at right angles. Show that it is possible in this case to turn over the four corners so that the vertices may meet in a point, and the area be everywhere doubled.

70. ABCD is a quadrilateral and ABCE is a parallelogram. Prove that $\triangle BCD = \triangle ACD \pm \triangle CDE$, according as A and E are on opposite sides or on the same side of CD.

71. ABCD, EBCF are two parallelograms on the same base and between the same parallels. If AC, BF meet in G, and BE, CD in H; show that GH (produced if necessary) passes through the mid-points of BC and DE.

72. ABCD, AHFE are two equal parallelograms having the $\angle BAD$ identical with the $\angle HAE$; show that $BE \parallel CF \parallel HD$.

73. With the same data, if BC, EF (produced if necessary) meet in O, and DC, HF in P; show that the points A, O and P are in one straight line.

74. ABC is a triangle, P is any point in AB, and the ▱ PBCR is completed. If PR meet AC in Q, show that $\triangle APR = \triangle ABQ$.

75. ABC is a triangle, right-angled at A; squares ABFG, ACKH are described externally to the triangle on the sides AB, AC; BK, CF meet AC, AB at N, M; show that $\triangle AHN = \triangle BCN$, and $\triangle AGM = \triangle BCM$.

76. With the same data as in the previous Ex., show that

$$AB^2 \sim AC^2 = CF^2 \sim BK^2.$$

77. Show that the sum of the squares on the lines joining any point in its plane to the vertices of a rectangle is double the sum of the squares on the perpendiculars drawn from that point to the sides of the rectangle.

78. In the $\triangle ABC$, D, E, F are points in BC, CA, AB such that

$$AF^2 + BD^2 + CE^2 = AE^2 + BF^2 + CD^2;$$

show that the perpendiculars drawn to the sides of the triangle through D, E, F are concurrent. *(This is the converse to* § 20, *Ex.* 1.)

79. P, Q, R are the feet of the altitudes of the $\triangle ABC$, $AP' \perp QR$, $BQ' \perp RP$, $CR' \perp PQ$; show that AP′, BQ′, CR′ are concurrent. *(This may be shown to depend on the previous Deduction.)*

80. If the perpendiculars from the vertices of the $\triangle ABC$ on the sides of a $\triangle PQR$ are concurrent, show that the perpendiculars from the vertices of the $\triangle PQR$ on the sides of the $\triangle ABC$ are also concurrent.

81. D, E, F are the mid-points of the sides of the △ ABC; OP, OQ, OR are three perpendiculars to the sides from a point O; P′, Q′, R′ are points on BC, CA, AB such that DP = DP′, etc.; show that the perpendiculars drawn to the sides of the △ ABC through P′, Q′, R′ are concurrent.

82. ABCD is a square whose diagonals intersect in O, and P is a point such that OP = OA; show that $PA^2+PB^2+PC^2+PD^2=4\,AB^2$.

83. P and Q are points on the same side of a straight line AB, R is the image of P in AB; (*See* § 21, *Ex.* 1, *Note*); show that $PQ<QR$.

84. D, E, F are the images of A, B, C in a given straight line; show that △ DEF = △ ABC.

85. If from the extremities of the base of a triangle straight lines are drawn to any point in the bisector of the exterior vertical angle, their sum shall be greater than the sum of the two sides.

86. Of all triangles which have the same vertical angle and the same altitude, show that the isosceles has the smallest area and smallest base.

87. Of all rectilineal figures which have the same area and the same number of sides, show that the equilateral has the least perimeter.

88. Of all rectilineal figures which have the same perimeter and the same number of sides, show that the equilateral has the greatest area.

89. ABCDE is a pentagon in which AB = BC = CD = DE, and ∠B = ∠D = right angle. If O be the mid-point of AE, and BO, OD bejoined, prove that ∠BOD = right angle, and that the three parts into which the pentagon is divided can be put together so as to form a square.

90. On each side of a triangle equilateral triangles are described outwards; show that their orthocentres are the vertices of an equilateral triangle.

§§ 22-30. EUCLID, I.

Problems.

91. AX and AY are two straight lines, through A draw a straight line PQ such that $\angle PAX = \angle QAY$.

92. On a given straight line construct a regular hexagon.

93. Trisect an angle which is equal to (1) one-half, (2) one-fourth of a right angle.

94. Find the position of a point such that the sum of its distances from the vertices of a quadrilateral may be a minimum.

95. ABC is a triangle and D, E, F are the mid-points of its sides; find the position of a point such that the sum of its distances from the six points A, B, C, D, E, F may be a minimum.

96. Through a given point A, draw three straight lines AP, AQ, AR such that they may be of given lengths, which satisfy the condition $AP+AR>2AQ$, and that Q may be the mid-point of PR.

Loci.

97. ABCD is a parallelogram of constant area on a given base BC. Find the locus of the mid-point of AB.

98. Find the locus of the point of intersection of the diagonals of a parallelogram of given area described upon a given base.

99. ABC is a triangle and P is any point in BC; PQ, PR, are drawn parallel to AB, AC, meeting AC, AB in Q and R; AP, QR meet in O; find the locus of O.

100. ABC is a triangle of constant area on a fixed base BC; its medians intersect in G; find the locus of G.

101. Find the locus of a point at which two equal segments of a given straight line subtend equal angles.

102. Find the locus of a point, the sum of whose distances from two parallel straight lines is equal to a given straight line greater than the distance between the parallel straight lines.

103. Find the locus of a point, the difference of whose distances from two parallel straight lines is equal to a given straight line less than the distance between the parallel straight lines.

104. Find the locus of a point, the sum of whose distances from two parallel straight lines is equal to the distance between the parallel straight lines.

105. Find the locus of a point, the difference of whose distances from two parallel straight lines is equal to the distance between the parallel straight lines.

106. Find the locus of a point, the sum of whose distances from two intersecting straight lines is equal to a given straight line.

107. Find the locus of a point, the difference of whose distances from two intersecting straight lines is equal to a given straight line.

108. A is a fixed point and XY a fixed straight line; P is any point in XY and PQ is drawn so that ∠APX = ∠QPY, and so that AP+PQ = a given straight line; find the locus of Q.

109. The lengths of the sides of a parallelogram are given, and one side is fixed in position; find the locus of the point of intersection of the diagonals of the parallelogram.

110. ABC is a triangle; PQ ∥ BC meets AB, AC in P, Q; BQ, CP intersect in O; find the locus of O.

111. The base BC of the △ ABC is fixed, and the median from B is of constant length; find the locus of A.

112. Find the locus of the vertex of a right-angled triangle whose hypotenuse is a given straight line.

113. OX, OY are two straight lines given in position and at right angles to one another; A, B are points in OX, OY, such that OA^2+OB^2 is constant; find the locus of the mid-point of AB.

114. On a fixed straight line AB, a △ APB is described, right-angled at P; Q, R are the mid-points of AP, BP, and S is the mid-point of QR; find the locus of S.

115. ABCD is a parallelogram, and P is a point within the ∠ ABD; if the difference between the △s PBC and PBD be constant, find the locus of P.

116. P is a point within a ▱ ABCD, through P straight lines are drawn parallel to the sides of the parallelogram; if the difference between the parallelograms PA, PC remain constant, find the locus of P.

Analysis and Synthesis, etc.

117. A, B, C, D are fixed points; find a point equidistant from A and B, such that the sum of its distances from C and D shall be a minimum.

118. AX, AY are two straight lines; draw a straight line BC meeting AX, AY in B, C, so that BC shall be equal and parallel to a given straight line.

119. AX, AY are two straight lines; draw a straight line BC meeting AX, AY in B, C, so that BC shall be equal to a given straight line, and shall make equal angles with AX, AY.

120. A, B, C are three given points; through A draw a straight line PQ, such that the distances from A to the feet of the perpendiculars drawn to PQ from B and C may be equal.

121. Find a point P in the hypotenuse BC of the right-angled triangle ABC, such that if PQ be drawn perpendicular to AC meeting it in Q, BP=PQ.

122. AX, AY are two given straight lines and P is a given point. Through P draw a straight line meeting AX and AY in Q and R, so that

(1) AQ=AR,
(2) AQ=QR.

123. ABC is a triangle, find points P, Q in AB, AC, so that

(1) $PQ \parallel BC$ and AP=CQ,
(2) $PQ \parallel BC$ and PQ=CQ,
(3) PQ=AP=CQ.

124. AB, AC are two given straight lines which form an acute angle, and P is a point within the angle. Find points Q and R in AB and AC, such that PQ, QR make equal angles with AB, and QR, RP make equal angles with AC. Show that your construction makes PQ+QR+RP a minimum.

125. E and F are two given points within the △ ABD of the rectangle ABCD; find four points P, Q, R, S in AB, BC, CD, DA, such that the rectilineal path EPQRSF may be a minimum.

126. E is a point within the △ ABD of the rectangle ABCD; find four points P, Q, R, S in AB, BC, CD, DA, such that the rectilineal path EPQRSE may be a minimum, and show that the length of the path is independent of the position of E.

Construction of Triangles, etc.

127. Construct an isosceles triangle which shall have the vertical angle equal to four times an angle at the base.

128. ABCD is a parallelogram and E is a point in AB; construct on AE as base, and with EAD as one of its angles, a parallelogram equal to ABCD.

129. Construct an isosceles triangle which shall have one-third of each of the angles at the base equal to half the vertical angle.

130. Construct a right-angled triangle, having given its perimeter and one of the acute angles.

131. Construct a $\triangle ABC$, having given
(1) AB, AC and the altitude from A;
(2) $\angle B$, $\angle C$ and the altitude from A;
(3) $\angle B$, BC and the median from A;
(4) BC, CA and the angle between BC and the median from A.

132. Construct a triangle, having given the perimeter, the altitude from the vertex, and one angle at the base.

133. Construct a $\triangle ABC$, having given $BC+CA-AB$, $\angle B$, $\angle C$. (*See Miscellaneous Deduction* 27, *page* 186.)

134. Construct a $\triangle ABC$, in which AN is the altitude from A, and AK is the bisector of the $\angle A$, having given
(1) BN, BK, $\angle B$;
(2) BK, CK, $\angle B$;
(3) CK, $\angle B$, $\angle C$;
(4) BK, CK, $\angle B \sim \angle C$;
(5) AK, BK, $\angle B \sim \angle C$;
(6) BN, BK, $\frac{1}{2}\angle A + \angle B$;
(7) AB, AK, $\frac{1}{2}\angle A + \angle C$.

135. Construct a rectangle ABCD, having given
(1) AB and AC;
(2) $AB+BC$ and $AB-BC$;
(3) AC and $AB+BC$;
(4) AC and $AB \sim BC$;
(5) AB and $AC+BC$;
(6) AB and $AC-BC$.

136. Construct a ▱ $ABCD$, whose diagonals intersect in O, having given

(1) AB, $AC+BC$, $\angle A$;
(2) AB, $AC \sim BC$, $\angle A$;
(3) AC, $AB+BC$, $\angle A$;
(4) AC, $AB-BC$, $\angle A$;
(5) AB, $AC+BD$, $\angle AOB$;
(6) AB, $AC \sim BD$, $\angle AOB$.

137. Construct a rhombus $ABCD$, having given

(1) AB and AC;
(2) AB and $AC+BD$;
(3) AB and $AC-BD$;
(4) $AC+BD$ and $\angle A$;
(5) $AB+AC$ and $\angle A$.

138. Construct a parallelogram, such that two of the sides shall lie on two given parallel straight lines, the other two sides shall pass through two given points, and the area shall equal a given area.

139. If A be a given point, BC and DE two given parallel straight lines, construct a square $APQR$, so that

(1) P and Q may lie on BC and DE;
(2) P and R may lie on BC and DE.

140. Construct a trapezium, having given one vertex, the mid-points of the diagonals, and the mid-point of one of the parallel sides.

141. Construct a quadrilateral, having given two opposite sides, the diagonals, and the angle contained by the diagonals.

142. Inscribe a square in the space enclosed by two equal intersecting circles.

143. In a given square inscribe a square whose area shall be three-fourths of the area of the given square.

144. Describe a square whose four sides shall pass respectively through four given points.

Show that, in general, six squares may be so described if the four points may be taken in any order.

Consider separately the cases in which the four points form a parallelogram, a rectangle, a rhombus, and a square.

§§ 31-41. EUCLID II.

Theorems.

145. ABC is a triangle, right-angled at A, and AC or CA is produced to E, so that $AC \cdot AE = AB^2$; show that $AC \cdot CE = BC^2$.

146. If P be any point within a $\triangle ABC$, and PL, PM, PN be drawn $\perp$ BC, CA, AB; show that

$$BC^2 + CA^2 + AB^2 = 2(BC \cdot BL + CA \cdot CM + AB \cdot AN).$$

147. A, B, C, D, E are five points in order on the same straight line, and $AB = CD$; show that $AE \cdot DE + BD \cdot CD = BE \cdot CE$.

148. If a straight line be divided into any number of segments, the square on the whole line is equal to the squares on the segments, together with twice the rectangles contained by each pair of segments.

149. K is any point on the diagonal AC of a rectangle $ABCD$; EKF parallel to BC meets AB, CD in E, F, and GKH parallel to AB meets AD, BC in G, H; show that

$$AK \cdot KC = EK \cdot KF + GK \cdot KH.$$

150. A, B, C, D are four points in order in the same straight line; show that $AD^2 + BC^2 = AC^2 + 2AB \cdot CD + BD^2$.

151. A, B, C, D, E are five points in order in the same straight line, and $AB = BC$, $CD = DE$; show that $AD^2 + 3DE^2 = BE^2 + 3AB^2$.

152. ABC is a triangle, right-angled at A, and $AN \perp BC$; show that

$$(BC + AN)^2 = (AB + AC)^2 + AN^2.$$

153. ABC is a triangle, right-angled at A; show that

$$(BC + CA + AB)^2 = 2\,(BC + CA)\,(AB + BC).$$

154. $ABCD$ is a square; P and Q are two points in BD; show that

$$AP^2 \sim AQ^2 = PQ\,(PB \sim QD).$$

155. Of all parallelograms with equal perimeters, the square has the greatest area.

156. $ABCD$ is a square; O is the mid-point of BD and P is any other point in BD; OM, PN are perpendicular to AB; show that

$$AM^2 > AN \cdot PN \text{ and that } \triangle AOM > \triangle APN.$$

157. A, B, C, D are four points in order in the same straight line, and $AB = BC$; show that $AD^2 - BD^2 = 3AB^2 + 2AB \cdot CD$.

158. ABCD is a quadrilateral, right-angled at B and D. If $AB+AD > BC+CD$, show that $AB \sim AD < BC \sim CD$.

159. In the $\triangle ABC$, AN is the altitude from A, and P is any point in AN; show that $AB \sim AC < PB \sim PC$.

160. In the $\triangle ABC$, AK is the bisector of the $\angle A$, and P is any point in AK; show that

$$AB \sim AC > PB \sim PC.$$

161. ABCD is a crossed quadrilateral in which $AB \parallel CD$ and $AD \perp BC$; E, F, G, H are the mid-points of AB, CD, AD, BC; show that

$$AB^2 \sim CD^2 = 4EF \cdot GH.$$

162. AB is bisected in O and divided in P; show that

$$AP^2+PB^2=2\,AP \cdot PB+4\,OP^2.$$

163. In the figure of § 36, Ex. 1, show that

(1) $AH > BH,\ H'A > AB$;
(2) $AB^2=H'A \cdot AH=H'B \cdot HB$;
(3) $AB^2=2HB^2+3AH \cdot HB=2H'B^2-3H'A \cdot H'B$;
(4) $(AB+BH)^2=5AH^2$; $(AB+H'B)^2=5H'A^2$.

164. AB is divided in H, so that $AB \cdot HB=AH^2$, and produced to K, so that $BK=AH$; show that $AK \cdot BK=AB^2$.

165. ABC is a triangle, right-angled at A, and AN is perpendicular to BC. If $AC=BN$, prove that BC is divided in medial section.

166. If, in the figure of § 36, Ex. 1, an equilateral triangle PAH be described, show that $PB=AG$.

167. In any right-angled triangle, twice the sum of the squares on the medians is equal to three times the square on the hypotenuse.

168. The base BC of a $\triangle ABC$ is trisected in D and E; show that

$$AB^2+AC^2=AD^2+AE^2+4DE^2.$$

169. ABC is a triangle, right-angled at A, and BC is trisected in D and E; show that

$$3(DE^2+EA^2+AD^2)=2BC^2.$$

170. In $\triangle ABC$, $AB > AC$, BC is trisected in D and E; show that

$$AB^2-AC^2=3\,(AD^2-AE^2).$$

171. ABCD is a quadrilateral; F, H are the mid-points of BC, DA, and I, J are the mid-points of AC, BD; show that

$$AB^2+CD^2=2\,FH^2+2\,IJ^2.$$

172. A is a given point and BX a given straight line; P and Q are two points in BX such that $AP^2+BP^2=AQ^2+BQ^2$; if C be the mid-point of AB, show that $CP=CQ$.

173. Show that the triangle is acute-angled whose sides measure 50, 41, and 39 inches respectively, and find the length of the altitude drawn to the shortest side.

174. Show that the triangle is obtuse-angled, whose sides measure 52, 33, and 25 inches respectively, and find the length of the altitude drawn to the intermediate side.

175. A, B, C, D are four points in order on the same straight line, E is the mid-point of AB, G of CD, and O of EG; if P be any other point in the line, show that

$$PA^2+PB^2+PC^2+PD^2=OA^2+OB^2+OC^2+OD^2+4PO^2.$$

176. E, F, G, H are the mid-points of AB, BC, CD, DA, the sides of a quadrilateral ABCD; show that

$$AB^2+CD^2+2EG^2=BC^2+DA^2+2FH^2.$$

177. In the $\triangle ABC$, E is the mid-point of AC and $AN \perp BC$; show that $BC \cdot BN=BE^2 \sim CE^2$.

178. G is the point of intersection of the medians of the $\triangle ABC$, and P is any other point, show that

$$3(PA^2+PB^2+PC^2)=BC^2+CA^2+AB^2+9PG^2.$$

179. ABCD is a quadrilateral; E is the mid-point of AB, G of CD O of EG and P is any point; show that

$$PA^2+PB^2+PC^2+PD^2=OA^2+OB^2+OC^2+OD^2+4PO^2.$$

180. ABC is a triangle right-angled at A, and $BC=2AC$; D is a point in AC produced, such that $CD=BC$; show that $BD^2=3BC^2$.

181. ABC is an equilateral triangle in which BC is produced to D, so that $CD=2BC$; show that $AD^2=7AB^2$.

§§ 39-41. EUCLID II.

Problems.

182. Construct a square equal to one-third of a given square.

183. Construct a rectangle equal to a given regular hexagon.

184. Construct a rectangle equal to a given regular octagon.

185. Construct two lines, having given their sum and the area of the rectangle they contain.

186. Construct a triangle, having given the base, the median to the base, and the difference of the squares on the two sides.

187. Divide a given straight line so that the sum of the squares on the segments may equal a given square.

188. If C be a given point in AB, find a point P in AB produced such that

$$AP^2+PB^2=2AC^2+2CB^2.$$

189. A, B, C, D are four points; find the locus of a fifth point P such that $PA^2+PB^2+PC^2+PD^2$ is constant.

190. A, B, C, D are four points; find the locus of P such that

$$PA^2+PB^2=PC^2+PD^2.$$

191. Find two straight lines, having given the sum and difference of their squares.

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